NONSTEADY HEAT CONDUCTION IN A LAYERED MEDIUM

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The process of nonsteady heat transfer in a plane multilayer body is investigated for boundary conditions of the third kind. The roots of the characteristic equation of the problem are determined by successive approximations.

The problem of the nonsteady temperature field in a composite body consisting of layers of materials with different properties is formulated as follows: the differential equation of heat conduction

$$\frac{\partial \vartheta_i}{\partial \mathrm{Fo}} = \frac{a_i}{a_1}\frac{\partial^2 \vartheta_i}{\partial X^2} \quad (i = 1,\ 2,\ \ldots,\ m); \tag{1}$$

the equality of the temperatures and heat fluxes at the boundaries of the layers

$$\vartheta_i = \vartheta_{i+1}, \tag{2}$$

$$\frac{\lambda_i}{\lambda_{i+1}}\frac{\partial \vartheta_i}{\partial X} = \frac{\partial \vartheta_{i+1}}{\partial X}; \tag{3}$$

the boundary conditions at the outer surfaces

$$\frac{\partial \vartheta_1(0,\ \mathrm{Fo})}{\partial X} = 0; \tag{4}$$

$$\frac{\partial \vartheta_m(p_m,\ \mathrm{Fo})}{\partial X} = -\mathrm{Bi}\,\vartheta_m(p_m,\ \mathrm{Fo});$$

$$p_i = \sum_{j=1}^{i} q_j, \quad q_i = \frac{\delta_i}{\delta_1}; \tag{5}$$

and the initial condition

$$\vartheta_i(X,\ 0) = 1. \tag{6}$$

In principle, the solution of system (1)–(6) presents no mathematical difficulties. The integral of problem (1)–(6) can be obtained on the basis of existing analytic methods of the theory of heat conduction [1]. However, at $m > 1$ the final results are so cumbersome [2] that they are difficult to use in practical calculations. Moreover, the complexity of the calculations and the end formulas increases sharply with increase in the number m of layers forming the body.

In many cases the method of successive approximations, proposed below for investigating system (1)–(6), makes possible the effective determination of the required temperature field $\vartheta_i(X, \mathrm{Fo})$.

The general solution of Eq. (1) can be represented in the form [2]

$$\vartheta_i(X,\ \mathrm{Fo}) = \sum_{n=1}^{\infty} (A_{n,i}\cos\mu_n h_i X + B_{n,i}\sin\mu_n h_i X)\exp(-\mu_n^2 \mathrm{Fo}), \tag{7}$$

where $h_i = (a_1/a_i)^{1/2}$.

It follows from condition (4) that the constant coefficients $B_{n,1} = 0$; i.e.,

$$\vartheta_1(X,\ \mathrm{Fo}) = \sum_{n=1}^{\infty} A_{n,1}\cos\mu_n X\exp(-\mu_n^2 \mathrm{Fo}). \tag{7'}$$

If relation (7) is substituted into conditions (2) and (3), we obtain a system of 2(m − 1) linear algebraic equations, solving which we find $A_{n,i+1}$ and $B_{n,i+1}$:

$$A_{n,i+1} = a_{n,i}A_{n,i}, \tag{8}$$

$$B_{n,i+1} = b_{n,i}A_{n,i}. \tag{9}$$

Here,

$$a_{n,i} = U_{n,i} + k_i W_{n,i} + c_{n,i}(P_{n,i} - k_i Q_{n,i}); \tag{10}$$

$$b_{n,i} = Q_{n,i} - k_i P_{n,i} + c_{n,i}(W_{n,i} + k_i U_{n,i}), \tag{11}$$

where

$$U_{n,i} = \cos\mu_n h_i p_i \cos\mu_n h_{i+1} p_i;$$

$$W_{n,i} = \sin\mu_n h_i p_i \sin\mu_n h_{i+1} p_i;$$

$$P_{n,i} = \sin\mu_n h_i p_i \cos\mu_n h_{i+1} p_i;$$

$$Q_{n,i} = \cos\mu_n h_i p_i \sin\mu_n h_{i+1} p_i;$$

$$k_i = \frac{\lambda_i}{\lambda_{i+1}}\sqrt{\frac{a_{i+1}}{a_i}}; \quad c_{n,i} = \frac{B_{n,i}}{A_{n,i}}; \quad c_{n,1} = 0.$$

Obviously, the coefficients $A_{n,i+1}$ and $B_{n,i+1}$ can be successively expressed in terms of $A_{n,1}$:

$$A_{n,i+1} = A_{n,1}\prod_{j=0}^{i} a_{n,j}, \quad a_{n,0} = 1, \tag{8'}$$

$$B_{n,i+1} = A_{n,1} b_{n,i}\prod_{j=0}^{i-1} a_{n,j}. \tag{9'}$$

The constants $A_{n,1}$ must be so defined that solution (7) is consistent with the initial condition (6); i.e.,

$$\vartheta_i(X,\ 0) = \sum_{n=1}^{\infty} A_{n,1}\varphi_{n,i}(\mu_n h_i X), \tag{12}$$

where

$$\varphi_{n,i} = \prod_{j=0}^{i-1} a_{n,j}\cos\mu_n h_i X +$$

$$+ b_{n,i-1} \prod_{j=0}^{i-2} a_{n,j} \sin \mu_n h_i X, \; b_{n,0} = 0.$$

The functions $\vartheta_i(X, 0)$ satisfy the Dirichlet conditions and therefore can be expanded in series (12). Multiplying the left- and right-hand sides of (12) by $\lambda_i \varphi_{k,i}/a_i$, integrating over the volume of each layer, and summing over all the layers, as a result of the orthogonality property of the eigenfunctions [2, 3]

$$\sum_{i=1}^{m} \frac{\lambda_i}{a_i} \int_{V_i} \varphi_{n,i} \varphi_{k,i} \, dV = 0 \text{ when } n \neq k,$$

we obtain the following explicit formula for finding $A_{n,1}$:

$$A_{n,1} = \frac{\sum_{i=1}^{m} \frac{\lambda_i}{a_i} \int_{p_{i-1}}^{p_i} \varphi_{n,i} \, dX}{\sum_{i=1}^{m} \frac{\lambda_i}{a_i} \int_{p_{i-1}}^{p_i} \varphi_{n,i}^2 \, dX}, \quad p_0 = 0.$$

After integration, we have

$$A_{n,1} = 2 \frac{N_n}{M_n}, \tag{13}$$

where

$$N_n = \sum_{i=1}^{m} \frac{\lambda_i}{a_i h_i} \left[ \prod_{j=0}^{i-1} a_{n,j} (\sin \nu_{n,i} - \sin \beta_{n,i}) - \right.$$

$$\left. - b_{n,i-1} \prod_{j=0}^{i-2} a_{n,j} (\cos \nu_{n,i} - \cos \beta_{n,i}) \right],$$

$$\nu_{n,i} = \mu_n h_i p_i; \quad \beta_{n,i} = \mu_n h_i p_{i-1};$$

$$M_n = \sum_{i=1}^{m} \frac{\lambda_i}{a_i h_i} \left[ \left( \prod_{j=0}^{i-1} a_{n,j}^2 + \right. \right.$$

$$\left. + b_{n,i-1}^2 \prod_{j=0}^{i-2} a_{n,j}^2 \right) (\nu_{n,i} - \beta_{n,i}) +$$

$$+ \frac{1}{2} \left( \prod_{j=0}^{i-1} a_{n,j}^2 - b_{n,i-1}^2 \prod_{j=0}^{i-2} a_{n,j}^2 \right) \times$$

$$\times (\sin 2\nu_{n,i} - \sin 2\beta_{n,i}) + 2 a_{n,i-1} b_{n,i-1} \times$$

$$\left. \times \prod_{j=0}^{i-2} a_{n,j}^2 (\sin^2 \nu_{n,i} - \sin^2 \beta_{n,i}) \right].$$

If

$$h_m \mu_n (\operatorname{tg} \mu_n h_m p_m - c_{n,m}) = \mathrm{Bi} (1 + c_{n,m} \operatorname{tg} \mu_n h_m p_m), \tag{14}$$

solution (7) will satisfy boundary condition (5). If we expand Eq. (14), substituting for the coefficient $c_{n,m}$ the expression

$$c_{n,m} = \frac{b_{n,m-1}}{a_{n,m-1}}, \tag{15}$$

where $a_{n,m-1}$ and $b_{n,m-1}$ are calculated from (10) and (11), we find that in the general case the roots $\mu_n$ are a function of $(2m - 1)$ parameters. Therefore, it is very difficult to represent the transcendental equation (14) in graphical or tabular form when $m > 1$. The chief difficulty in calculating the temperature field described by system (1)–(6) also consists in determining $\mu_n$.

Let us use an iteration method to find $\mu_n$. To this end, we write (14) in the form

$$\nu_n (\operatorname{tg} \nu_n - c_{n,m}) = \mathrm{Bi}^* (1 + c_{n,m} \operatorname{tg} \nu_n). \tag{14'}$$

Here,

$$\nu_n = \mu_n h_m p_m; \quad \mathrm{Bi}^* = \mathrm{Bi}\, p_m.$$

The first approximation for $\nu_n$ can be obtained by taking $c_{n,m} = 0$. Then

$$\nu_n^{(1)} \operatorname{tg} \nu_n^{(1)} = \mathrm{Bi}^*. \tag{16}$$

The values of the first six roots of Eq. (16) are presented in the monograph [1].

Using $\nu_n^{(1)}$, from (15) we compute the first iteration for $c_{n,m}^{(1)}$. We then find the new, more accurate value of $\nu_n$ from the relation

$$\nu_n^{(2)} (\operatorname{tg} \nu_n^{(2)} - c_{n,m}^{(1)}) = \mathrm{Bi}^* (1 + c_{n,m}^{(1)} \operatorname{tg} \nu_n^{(2)}). \tag{17}$$

Then, using $\nu_n^{(2)}$, we calculate the next iteration for $c_{n,m}$ from expression (15).

By repeating the process several times, we can obtain the value of the root $\nu_n$ with any degree of accuracy. To simplify the operation of determining $\nu_n$, it is convenient to tabulate relation (14') or represent it in graphical form (figure). The abscissas of the points of intersection of the curves

$$y_1 = \frac{1 + c \operatorname{tg} \nu}{\operatorname{tg} \nu - c}$$

and the straight lines

$$y_2 = \frac{\nu}{\mathrm{Bi}^*}$$

give the values of $\nu_1$ and $\nu_2$. The graphs presented in the figure can be used to calculate the temperature field of a body with any number of layers.

The complexes $U_{n,i}$, $W_{n,i}$, $P_{n,i}$ and $Q_{n,i}$ can also be represented in the form of tables (Tables 1 and 2), the use of which accelerates the process of finding $a_{n,i}$ and $b_{n,i}$.

As an example, Table 3 presents the results of a computation of the first two roots $\mu_1$ and $\mu_2$ for a two-layer system. The same table includes values of these roots obtained by graphical solution of the characteristic equation

$$\mu h (k_1 \operatorname{tg} \mu + \operatorname{tg} \mu h q) = \mathrm{Bi} (1 - k_1 \operatorname{tg} \mu \operatorname{tg} \mu h q),$$

$$k_1 = \frac{\lambda_1}{\lambda_2} \sqrt{\frac{a_2}{a_1}}; \quad h = \sqrt{\frac{a_1}{a_2}}; \quad q = \frac{\delta_2}{\delta_1}, \tag{18}$$

which is valid in the case of a two-layer body [1].

## Table 1

### Values of the Functions U = cos $\nu$ cos r$\nu$ and W = sin $\nu$ sin r$\nu$

| $\nu$ | r = 0.1 U | 0.1 W | 0.2 U | 0.2 W | 0.3 U | 0.3 W | 0.4 U | 0.4 W | 0.5 U | 0.5 W |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.0000 | 0.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 |
| 0.1 | 0.9950 | 0.0010 | 0.9948 | 0.0020 | 0.9946 | 0.0030 | 0.9942 | 0.0040 | 0.9938 | 0.0050 |
| 0.2 | 0.9799 | 0.0040 | 0.9793 | 0.0079 | 0.9783 | 0.0119 | 0.9769 | 0.0159 | 0.9752 | 0.0198 |
| 0.3 | 0.9549 | 0.0089 | 0.9536 | 0.0177 | 0.9515 | 0.0266 | 0.9485 | 0.0354 | 0.9446 | 0.0442 |
| 0.4 | 0.9203 | 0.0156 | 0.9181 | 0.0311 | 0.9144 | 0.0466 | 0.9093 | 0.0620 | 0.9027 | 0.0774 |
| 0.5 | 0.8765 | 0.0240 | 0.8732 | 0.0479 | 0.8677 | 0.0716 | 0.8601 | 0.0952 | 0.8503 | 0.1186 |
| 0.6 | 0.8239 | 0.0339 | 0.8194 | 0.0676 | 0.8120 | 0.1011 | 0.8017 | 0.1342 | 0.7885 | 0.1669 |
| 0.7 | 0.7630 | 0.0451 | 0.7574 | 0.0899 | 0.7480 | 0.1343 | 0.7351 | 0.1780 | 0.7185 | 0.2209 |
| 0.8 | 0.6945 | 0.0573 | 0.6878 | 0.1143 | 0.6767 | 0.1705 | 0.6613 | 0.2257 | 0.6417 | 0.2794 |
| 0.9 | 0.6191 | 0.0704 | 0.6116 | 0.1402 | 0.5991 | 0.2089 | 0.5818 | 0.2759 | 0.5597 | 0.3407 |
| 1.0 | 0.5376 | 0.0840 | 0.5295 | 0.1672 | 0.5162 | 0.2487 | 0.4976 | 0.3277 | 0.4742 | 0.4034 |
| 1.1 | 0.4509 | 0.0978 | 0.4427 | 0.1945 | 0.4291 | 0.2888 | 0.4104 | 0.3796 | 0.3867 | 0.4658 |
| 1.2 | 0.3598 | 0.1116 | 0.3520 | 0.2215 | 0.3391 | 0.3283 | 0.3214 | 0.4304 | 0.2991 | 0.5263 |
| 1.3 | 0.2652 | 0.1249 | 0.2585 | 0.2477 | 0.2474 | 0.3663 | 0.2321 | 0.4788 | 0.2130 | 0.5831 |
| 1.4 | 0.1683 | 0.1375 | 0.1634 | 0.2723 | 0.1552 | 0.4018 | 0.1440 | 0.5235 | 0.1300 | 0.6348 |
| 1.5 | 0.0699 | 0.1491 | 0.0676 | 0.2948 | 0.0637 | 0.4339 | 0.0584 | 0.5632 | 0.0518 | 0.6799 |
| 1.6 | −0.0288 | 0.1593 | −0.0277 | 0.3144 | −0.0259 | 0.4616 | −0.0234 | 0.5969 | −0.0203 | 0.7171 |
| 1.7 | −0.1270 | 0.1678 | −0.1215 | 0.3307 | −0.1124 | 0.4841 | −0.1002 | 0.6235 | −0.0850 | 0.7450 |
| 1.8 | −0.2235 | 0.1743 | −0.2126 | 0.3431 | −0.1949 | 0.5007 | −0.1708 | 0.6421 | −0.1412 | 0.7628 |
| 1.9 | −0.3175 | 0.1787 | −0.3002 | 0.3510 | −0.2722 | 0.5107 | −0.2343 | 0.6519 | −0.1881 | 0.7697 |
| 2.0 | −0.4078 | 0.1807 | −0.3833 | 0.3541 | −0.3[illegible]34 | 0.5134 | −0.2899 | 0.6523 | −0.2248 | 0.7651 |
| 2.5 | −0.7762 | 0.1481 | −0.7031 | 0.2869 | −0.5862 | 0.4079 | −0.4329 | 0.5036 | −0.2526 | 0.5679 |
| 3.0 | −0.9458 | 0.0417 | −0.8171 | 0.0797 | −0.6154 | 0.1105 | −0.3587 | 0.1315 | −0.0700 | 0.1408 |

| $\nu$ | r = 0.6 U | 0.6 W | 0.7 U | 0.7 W | 0.8 U | 0.8 W | 0.9 U | 0.9 W | 1.0 U | 1.0 W |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.0000 | 0.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 |
| 0.1 | 0.9932 | 0.0060 | 0.9926 | 0.0070 | 0.9918 | 0.0080 | 0.9910 | 0.0090 | 0.9900 | 0.0100 |
| 0.2 | 0.9730 | 0.0238 | 0.9705 | 0.0277 | 0.9676 | 0.0317 | 0.9642 | 0.0356 | 0.9605 | 0.0395 |
| 0.3 | 0.9390 | 0.0529 | 0.9344 | 0.0616 | 0.9280 | 0.0702 | 0.9207 | 0.0788 | 0.9127 | 0.0873 |
| 0.4 | 0.8947 | 0.0926 | 0.8852 | 0.1076 | 0.8743 | 0.1225 | 0.8620 | 0.1372 | 0.8484 | 0.1516 |
| 0.5 | 0.8384 | 0.1417 | 0.8244 | 0.1644 | 0.8083 | 0.1867 | 0.7902 | 0.2085 | 0.7701 | 0.2299 |
| 0.6 | 0.7724 | 0.1989 | 0.7536 | 0.2302 | 0.7321 | 0.2607 | 0.7079 | 0.2903 | 0.6812 | 0.3188 |
| 0.7 | 0.6984 | 0.2627 | 0.6748 | 0.3032 | 0.6480 | 0.3422 | 0.6180 | 0.3795 | 0.5850 | 0.4150 |
| 0.8 | 0.6180 | 0.3313 | 0.5903 | 0.3811 | 0.5588 | 0.4284 | 0.5238 | 0.4730 | 0.4854 | 0.5146 |
| 0.9 | 0.5332 | 0.4027 | 0.5023 | 0.4615 | 0.4673 | 0.5165 | 0.4286 | 0.5674 | 0.3864 | 0.6136 |
| 1.0 | 0.4459 | 0.4751 | 0.4132 | 0.5421 | 0.3764 | 0.6036 | 0.3359 | 0.6591 | 0.2919 | 0.7081 |
| 1.1 | 0.3583 | 0.5464 | 0.3256 | 0.6204 | 0.2890 | 0.6869 | 0.2489 | 0.7451 | 0.2058 | 0.7942 |
| 1.2 | 0.2724 | 0.6146 | 0.2419 | 0.6940 | 0.2078 | 0.7635 | 0.1708 | 0.8220 | 0.1313 | 0.8687 |
| 1.3 | 0.1902 | 0.6777 | 0.1642 | 0.7607 | 0.1354 | 0.8310 | 0.1044 | 0.8872 | 0.0716 | 0.9284 |
| 1.4 | 0.1134 | 0.7338 | 0.0947 | 0.8184 | 0.0741 | 0.8870 | 0.0520 | 0.9382 | 0.0289 | 0.9711 |
| 1.5 | 0.0440 | 0.7814 | 0.0352 | 0.8652 | 0.0256 | 0.9297 | 0.0155 | 0.9733 | 0.0050 | 0.9950 |
| 1.6 | −0.0167 | 0.8188 | −0.0127 | 0.8997 | −0.0084 | 0.9576 | −0.0038 | 0.9910 | 0.0009 | 0.9991 |
| 1.7 | −0.0674 | 0.8450 | −0.0479 | 0.9206 | −0.0270 | 0.9697 | −0.0053 | 0.9908 | 0.0166 | 0.9834 |
| 1.8 | −0.1071 | 0.8589 | −0.0695 | 0.9272 | −0.0296 | 0.9655 | 0.0112 | 0.9727 | 0.0516 | 0.9484 |
| 1.9 | −0.1350 | 0.8598 | −0.0771 | 0.9190 | −0.0164 | 0.9451 | 0.0449 | 0.9371 | 0.1045 | 0.8955 |
| 2.0 | −0.1508 | 0.8475 | −0.0707 | 0.8961 | 0.0122 | 0.9089 | 0.0945 | 0.8855 | 0.1732 | 0.8268 |
| 2.5 | −0.0567 | 0.5970 | −0.1428 | 0.5889 | 0.3334 | 0.5442 | 0.5033 | 0.4657 | 0.6418 | 0.3582 |
| 3.0 | 0.2249 | 0.1374 | −0.4998 | 0.1218 | 0.7300 | 0.0953 | 0.8950 | 0.0603 | 0.9801 | 0.0199 |

## Table 2

### Values of the Functions P = sin $\nu$ cos r$\nu$ and Q = cos $\nu$ sin r$\nu$

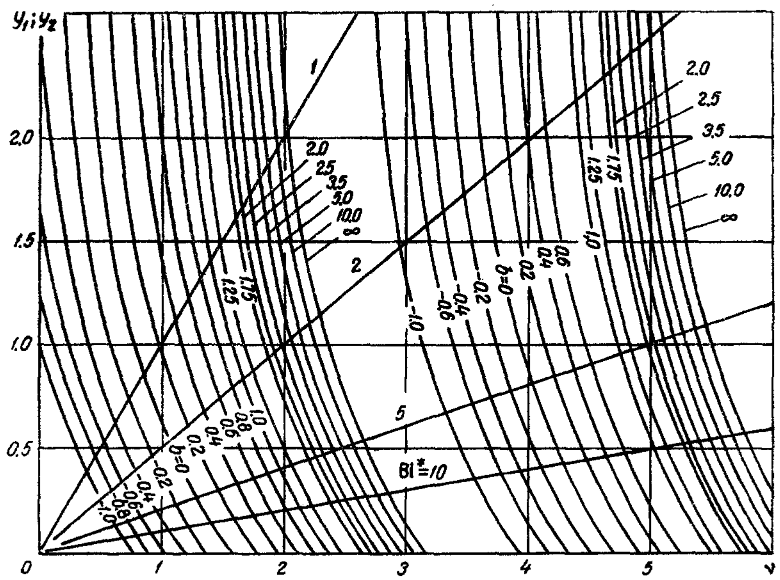
| $\nu$ | r = 0.1 P | 0.1 Q | 0.2 P | 0.2 Q | 0.3 P | 0.3 Q | 0.4 P | 0.4 Q | 0.5 P | 0.5 Q |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.1 | 0.0998 | 0.0100 | 0.0998 | 0.0199 | 0.0998 | 0.0299 | 0.0998 | 0.0398 | 0.0997 | 0.0497 |
| 0.2 | 0.1986 | 0.0196 | 0.1985 | 0.0392 | 0.1983 | 0.0588 | 0.1980 | 0.0783 | 0.1977 | 0.0978 |
| 0.3 | 0.2954 | 0.0287 | 0.2950 | 0.0573 | 0.2943 | 0.0859 | 0.2934 | 0.1144 | 0.2922 | 0.1428 |
| 0.4 | 0.3891 | 0.0368 | 0.3882 | 0.0736 | 0.3866 | 0.1103 | 0.3844 | 0.1467 | 0.3817 | 0.1830 |
| 0.5 | 0.4788 | 0.0439 | 0.4770 | 0.0876 | 0.4740 | 0.1311 | 0.4699 | 0.1743 | 0.4645 | 0.2171 |
| 0.6 | 0.5636 | 0.0495 | 0.5606 | 0.0988 | 0.5555 | 0.1478 | 0.5485 | 0.1962 | 0.5394 | 0.2439 |
| 0.7 | 0.6426 | 0.0535 | 0.6379 | 0.1067 | 0.6301 | 0.1594 | 0.6191 | 0.2114 | 0.6052 | 0.2623 |
| 0.8 | 0.7151 | 0.0557 | 0.7082 | 0.1110 | 0.6968 | 0.1656 | 0.6809 | 0.2192 | 0.6607 | 0.2713 |
| 0.9 | 0.7802 | 0.0559 | 0.7707 | 0.1113 | 0.7549 | 0.1658 | 0.7331 | 0.2190 | 0.7053 | 0.2704 |
| 1.0 | 0.8373 | 0.0539 | 0.8247 | 0.1073 | 0.8039 | 0.1597 | 0.7750 | 0.2104 | 0.7385 | 0.2590 |
| 1.1 | 0.8858 | 0.0498 | 0.8697 | 0.0990 | 0.8431 | 0.1470 | 0.8063 | 0.1932 | 0.7598 | 0.2371 |
| 1.2 | 0.9253 | 0.0434 | 0.9053 | 0.0861 | 0.8723 | 0.1276 | 0.8267 | 0.1673 | 0.7692 | 0.2046 |
| 1.3 | 0.9554 | 0.0347 | 0.9312 | 0.0688 | 0.8912 | 0.1017 | 0.8362 | 0.1329 | 0.7671 | 0.1619 |
| 1.4 | 0.9758 | 0.0237 | 0.9471 | 0.0470 | 0.8998 | 0.0693 | 0.8349 | 0.0903 | 0.7537 | 0.1095 |
| 1.5 | 0.9863 | 0.0106 | 0.9529 | 0.0209 | 0.8982 | 0.0308 | 0.8233 | 0.0399 | 0.7299 | 0.0482 |
| 1.6 | 0.9868 | −0.0047 | 0.9488 | −0.0092 | 0.8866 | −0.0135 | 0.8018 | −0.0174 | 0.6964 | −0.0209 |
| 1.7 | 0.9774 | −0.0218 | 0.9349 | −0.0430 | 0.8655 | −0.0629 | 0.7711 | −0.0810 | 0.6545 | −0.0968 |
| 1.8 | 0.9598 | −0.0407 | 0.9114 | −0.0800 | 0.8353 | −0.1168 | 0.7322 | −0.1498 | 0.6054 | −0.1780 |
| 1.9 | 0.9293 | −0.0611 | 0.8788 | −0.1199 | 0.7967 | −0.1745 | 0.6859 | −0.2227 | 0.5505 | −0.2630 |
| 2.0 | 0.8912 | −0.0827 | 0.8375 | −0.1620 | 0.7505 | −0.2350 | 0.6335 | −0.2985 | 0.4913 | −0.3502 |
| 2.5 | 0.5799 | −0.1982 | 0.5252 | −0.3841 | 0.4379 | −0.5461 | 0.3234 | −0.6741 | 0.1887 | −0.7603 |
| 3.0 | 0.1348 | −0.2926 | 0.1165 | −0.5590 | 0.0877 | −0.7755 | 0.0511 | −0.9227 | 0.0100 | −0.9875 |

| $\nu$ | r = 0.6 P | 0.6 Q | 0.7 P | 0.7 Q | 0.8 P | 0.8 Q | 0.9 P | 0.9 Q | 1.0 P = Q |
|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0 000 |
| 0.1 | 0.0997 | 0.0597 | 0.0996 | 0.0696 | 0.0995 | 0.0795 | 0.0994 | 0.0894 | 0.0 993 |
| 0.2 | 0.1972 | 0.1173 | 0.1967 | 0.1368 | 0.1961 | 0.1561 | 0.1955 | 0.1755 | 0.1 947 |
| 0.3 | 0.2907 | 0.1710 | 0.2890 | 0.1992 | 0.2871 | 0.2271 | 0.2848 | 0.2548 | 0.2 823 |
| 0.4 | 0.3783 | 0.2189 | 0.3743 | 0.2545 | 0.3697 | 0.2897 | 0.3645 | 0.3245 | 0.3 587 |
| 0.5 | 0.4580 | 0.2593 | 0.4504 | 0.3009 | 0.4416 | 0.3417 | 0.4317 | 0.3817 | 0.4 207 |
| 0.6 | 0.5284 | 0.2907 | 0.5156 | 0.3365 | 0.5008 | 0.3811 | 0.4843 | 0.4243 | 0.4 660 |
| 0.7 | 0.5882 | 0.3119 | 0.5684 | 0.3600 | 0.5458 | 0.4063 | 0.5205 | 0.4506 | 0.4 927 |
| 0.8 | 0.6363 | 0.3217 | 0.6078 | 0.3701 | 0.5754 | 0.4161 | 0.5393 | 0.4594 | 0.4 998 |
| 0.9 | 0.6719 | 0.3196 | 0.6330 | 0.3662 | 0.5889 | 0.4099 | 0.5401 | 0.4502 | 0.4869 |
| 1.0 | 0.6945 | 0.3051 | 0.6436 | 0.3481 | 0.5863 | 0.3876 | 0.5231 | 0.4232 | 0.4546 |
| 1.1 | 0.7040 | 0.2781 | 0.6398 | 0.3158 | 0.5678 | 0.3496 | 0.4890 | 0.3792 | 0.4043 |
| 1.2 | 0.7007 | 0.2389 | 0.6221 | 0.2698 | 0.5345 | 0.2968 | 0.4393 | 0.3196 | 0.3377 |
| 1.3 | 0.6850 | 0.1881 | 0.5914 | 0.2112 | 0.4878 | 0.2307 | 0.3759 | 0.2463 | 0.2578 |
| 1.4 | 0.6577 | 0.1266 | 0.5489 | 0.1412 | 0.4293 | 0.1530 | 0.3014 | 0.1618 | 0.1675 |
| 1.5 | 0.6200 | 0.0554 | 0.4963 | 0.0614 | 0.3615 | 0.0659 | 0.2185 | 0.0690 | 0.0706 |
| 1.6 | 0.5733 | −0.0239 | 0.4355 | −0.0263 | 0.2866 | −0.0280 | 0.1304 | −0.0290 | −0.0292 |
| 1.7 | 0.5190 | −0.1098 | 0.3686 | −0.1196 | 0.2075 | −0.1260 | 0.0404 | −0.1287 | −0.1278 |
| 1.8 | 0.4590 | −0.2004 | 0.2978 | −0.2163 | 0.1270 | −0.2253 | 0.0479 | −0.2269 | −0:2213 |
| 1.9 | 0.3952 | −0.2938 | 0.2257 | −0.3140 | 0.0480 | −0.3229 | −0.1313 | −0.3202 | −0.3059 |
| 2.0 | 0.3295 | −0.3878 | 0.1546 | −0.4101 | −0.0266 | −0.4160 | −0.2066 | −0.4052 | −0.3784 |
| 2.5 | 0.0423 | −0.7991 | 0.1017 | −0.7883 | −0.2490 | −0.7285 | −0.3759 | −0.6233 | −0.4795 |
| 3.0 | −0.0321 | −0.9641 | −0.0712 | −0.8546 | −0.1041 | −0.6687 | −0.1276 | −0.4231 | −0.1397 |

Table 3

Values of the First Two Roots of Characteristic Equation (14') for $Bi = 0.8,\ k_1 = 0.5,\ h = 2.0,\ q = 0.25,\ m = 2$

| No. of operation | $c_{1,m}$ acc. to (15) | $\mu_1$ acc. to (14') | $\mu_1$ acc. to (18) | $c_{2,m}$ acc. to (15) | $\mu_2$ acc. to (14') | $\mu_2$ acc. to (18) |
|---|---|---|---|---|---|---|
| 0 | 0 | 0.344 | | 0 | 1.370 | |
| 1 | 0.561 | 0.482 | | 66.894 | 1.960 | |
| 2 | 0.855 | 0.538 | 0.579 | —10.936 | 2.000 | 2.014 |
| 3 | 1.002 | 0.562 | | —8.474 | 2.010 | |
| 4 | 1.069 | 0.572 | | —7.996 | 2.013 | |



Geometric representation of characteristic equation (14').

NOTATION

$\vartheta_i(X, Fo) = T_m - T_i(X, Fo)/(T_m - T_0)$ is the dimensionless temperature of the i-th body; $T_m$ is the temperature of the medium; $T_0$ is the initial temperature of the system; $X = x/\delta_1$ is the dimensionless coordinate; $Fo = a_1\tau/\delta_1^2$ is the Fourier number; $Bi = \alpha\delta_1/\lambda_m$ is the Biot number; $\delta_i$ is the thickness of the i-th body; $\lambda_i$ and $a_i$ are the thermal conductivity and thermal diffusivity of the i-th body; $\alpha$ is the heat-transfer coefficient; $\tau$ is the time.

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